# Index Tracking by Means of Optimized Sampling

KEES VAN MONTFORT, ELOUT VISSER, AND LAURENS FIJN VAN DRAAT

**KEES VAN MONTFORT** is a professor at Vrije Universiteit Amsterdam and Nyenrode Business Universiteit in Amsterdam, The Netherlands.
**kvmontfort@feweb.vu.nl**

**ELOUT VISSER** is a researcher at Ortec in Amsterdam, The Netherlands.
**evisser@ortec.nl**

**LAURENS FIJN VAN DRAAT** is a researcher at Ortec in Gouda, The Netherlands.
**lfijnvandraat@ortec.nl**

Investors pursue active investment strategies if they are convinced the market is not efficient. They aim to obtain a higher return than the index.

Active strategies have several disadvantages. The first is that they entail high research and transaction costs, which diminish any returns gained. Second, active strategies also often entail high risks, making the return required higher.

We have seen in recent years growing interest in an objective way to value investment returns, or what is referred to as *performance measurement.* The focus shifts from the absolute return to the relative return, because one wants to draw a comparison with the return figures of colleagues or of a benchmark. With this shift in focus, active portfolios have become more risky than passive portfolios. Tracking an index then becomes a good option.

The third major disadvantage of active investment strategies is that they require considerable knowledge about stocks and markets. More and more investments are made in foreign countries with their diversification advantages accompanied by easier access to those markets. Yet investors often know relatively little about these new markets. Again index investment may be more attractive in this case.

With the lower costs of investing passively, the various disadvantages of active investing are important reasons to look at index tracking, or attempting to imitate the return on the index instead of beating it. In theory, the capital asset pricing model (CAPM) supports this strategy. A number of assumptions support the conclusion that the only rational investment strategy is to hold the market portfolio.

There are four different methods of index tracking. The most obvious method is *complete replication* (see Rudd [1980]). The portfolio that is supposed to track the index, the so-called index portfolio, consists of exactly the same stocks as the index and in the same proportions. This, of course, guarantees the same return as the index, excluding transaction costs.

Apart from transaction costs, the bid–ask spread (the difference between the buying and the selling price) also has an impact on accurate tracking of the index. When you buy stocks you pay the (higher) ask price, and when you sell you get the (lower) bid price. Liquid securities, mostly large-company stocks, have a narrow bid–ask spread, but for smaller stocks that are less traded the impact can be greater because they tend to have wider spreads.

Another way to compose an index portfolio is *synthetic indexing* (see Dynkin, Hyman, and Wu [1997]). In this case the index portfolio consists of a future on the index and the right amount of cash. As only the difference between the actual price and the fixed price is paid, futures transactions can be made at low cost. A disadvantage of synthetic indexing is that futures are available in only a limited number of indexes.

A third approach to index tracking is to use a *stratified index portfolio*, or divide the stocks of the index into a number of different categories such as sectors or countries (see Larsen, Bruce, and Resnick [1998], Focardi and Fabozzi [2004]), and Frino et al. [2004]. The index portfolio is then put together so that each category is represented in the index portfolio to the same extent as in the index (for example, choosing the stock with the highest market capitalization within each category and assigning weights of all the stocks from this category to this stock). The presupposition is that the price fluctuations of stocks in one category are closely linked, while the price fluctuations of stocks of different categories differ considerably.

The fourth method of index tracking, which we examine here more thoroughly, is *optimized sampling*. This approach does not differentiate between separate categories, and from a mathematical point of view can give better results than stratification. It aims to find the portfolio that has tracked the index as much as possible in the past, hoping it will track the index the same way in the future. The number of stocks in this index portfolio will be restricted on the one hand, but on the other, transaction costs will be limited (see Alexander and Dimitriu [2004], among others).

Although there is much more research on active investment strategies than on passive investment, there is some on index tracking in general and optimized sampling in particular. Bertsimas, Darnell, and Soucy [1999] develop a model seeking a minimal deviation between constitution of the index portfolio and the index. They add penalty terms to the number of stocks included, transaction costs, and weights per sector to try to avoid high management and transaction costs without losing the characteristics of the index.

A more common objective in optimized sampling is for the index portfolio to approach the return rather than the constitution of the index. In another approach based on the function of objectives, Meade and Salkin [1989] develop a model to calculate the index portfolio with minimal historical tracking error. Even though they conclude their model is the best, Meade and Salkin do not deal with optimal stock selection.

One may often want to limit the number of stocks that go to make up the index portfolio, certainly for tracking large indexes. Thus a good model should also be able to determine which stocks are in the index portfolio. Adding restrictions to the number of stocks will lead to additional binary variables, so finding the optimal solutions for these problems will soon take too much time to calculate.

This is why Tabata and Tekada [1995] develop heuristics that calculate a local optimal index portfolio. They apply these heuristics to a problem of limited size: 15 stocks, for which the index portfolio must consist of a fixed number of 3 stocks selected in advance.

Beasly, Meade and Chang [1999] opt for a totally different approach. They use generic algorithms to solve the problem of selecting stocks and determining weights. They also include a restriction to prevent the index portfolio from leading to very high transaction costs. After applying these heuristics to the tracking of a large number of indexes (with an index portfolio consisting of ten stocks), they conclude that the precise number of stocks included has an influence on the tracking error when only low transaction costs are permitted.

Jansen and Van Dijk [2002] deal with optimal benchmark tracking for small portfolios by solving some alternative minimization problems (not including transaction costs). This approach is also difficult to apply. Fund managers must choose a priori some optimal values for two model parameters. Simple rules for the choice of these values are not available.

We introduce a universally applicable model to determine an index portfolio, addressing the function of objectives, restrictions, selection of stocks, transaction costs, and the solution heuristics. Backtests of the model for the MSCI Europe Index over one year show that the performance of our optimized index portfolio differs only slightly from the index performance.

Our focus is primarily on practical use of the index portfolios constructed, by emphasizing the period after optimizing. This approach is different from that in Beasley, Meade, and Chang [1999], who focus principally on the quality of the optimizing and thereby emphasize the optimizing period itself. Our approach also requires fewer assumptions than the approach favored by Rudolf, Wolter, and Zimmermann [1999], who assume that the relation between the price of the chosen portfolio and the price of an arbitrary stock is constant throughout the entire optimization period.

## OBJECTIVE FUNCTION

"Correctly tracking" the index is usually measured by what is referred to as tracking error. Tracking error is defined in the literature in a number of different ways (see Kritzman [1987]; Roll [1992]; Speranza [1993]; Clarke, Krase, and Statman [1994]; Beasley, Meade, and Chang [1999], and Rudolf, Wolter, and Zimmermann [1999]). When $r_{I,t}$ and $r_{P,t}$ represent the daily return on day $t$ of the index and the index tracking portfolio, respectively, we use for the tracking error over the period [1, $T$]:

$$TE = \sum_{t=1}^{T} (r_{I,t} - r_{P,t})^2 \tag{1}$$

This is the sum of the squared differences in the daily returns of the index and the index portfolio.

It should be immediately clear from the definition that an index portfolio is good when it has low tracking error. We assume—and put this hypothesis to the test—that a portfolio with low tracking error in the recent past will also have low tracking error in the near future. To find the stocks and the corresponding weights for an index tracking portfolio with tracking error that is as low as possible, we formulate a mathematical programming problem. The input variables are defined as follows:

$N$ = number of available stocks;
$T$ = current time (when the optimization takes place);
$P_{i,t}$ = historical price of stock $i$ on day $t$ ($i = 1, \ldots, N$; $t = 0, \ldots, T$); and
$P_{P,t}$ = historical price of the index tracking portfolio on day $t$ ($t = 0, \ldots, T$).

The following auxiliary variables are defined as follows:

$A_i$ = number of shares of stock $i$ in the index portfolio to be constructed ($i = 1, \ldots, N$) (an integer variable);
$r_{i,t}$ = return of stock $i$ on day $t$ ($i = 1, \ldots, N$; $t = 1, \ldots, T$); and
$r_{P,t}$ = return of the index tracking portfolio on day $t$ ($t = 1, \ldots, T$).

The decision variable is:

$X_i$ = fraction of the value of the index portfolio that is invested at time $T$ in stock $i$ (for $i = 1, \ldots, N$).

Useful relations among these variables include:

$$P_{P,t} = \sum_{i=1}^{N} A_i P_{i,t} \tag{2}$$

$$A_i = \frac{X_i P_{P,T}}{P_{i,T}} \tag{3}$$

$r_{I,t}$ and $r_{P,t}$ can now be written as follows:

$$r_{I,t} = \frac{P_{I,t} - P_{I,t-1}}{P_{I,t-1}} \tag{4}$$

$$r_{P,t} = \frac{P_{P,t} - P_{P,t-1}}{P_{P,t-1}} = \sum_{i=1}^{N} \frac{A_i P_{i,t} - A_i P_{i,t-1}}{P_{P,t-1}} = \sum_{i=1}^{N} X_i \frac{P_{P,T}}{P_{i,T}} \left( \frac{P_{i,t} - P_{i,t-1}}{P_{P,t-1}} \right) \tag{5}$$

Note that in Equation (5) the decision variables $X_i$ are also present in the term $P_{P,t}$. Consequently, the relation between $r_{P,t}$ and $X_i$ is not linear, and the optimization of the objective function becomes complicated in that it will be *NP* difficult. This means there are no known methods to definitely solve the problem efficiently.

*Efficient* here means that computer time does not rise disproportionally for a higher number of stocks.

Rudolf, Wolter, and Zimmermann [1999] assume that:

$$\frac{P_{P,T}}{P_{i,T}} = \frac{P_{P,t}}{P_{i,t}} \quad \text{for all } t \text{ and } i$$

so that we get a linear expression for Equation (5). If we do not want to make this very restrictive assumption, we can use an other approach that also makes Equation (5) linear. Therefore we use an approach similar to that of Meade and Salkin [1989]. After all, for a good index portfolio the relation:

$$\frac{P_{P,T}}{P_{P,t-1}} \approx \frac{P_{I,T}}{P_{I,t-1}} \quad \text{(has to prevail for all } t\text{)} \tag{6}$$

By substituting (6) in (5) we can estimate $r_{P,t}$:

$$r_{P,t} \approx \sum_{i=1}^{N} X_i \frac{P_{I,T}}{P_{i,T}} \left( \frac{P_{i,t} - P_{i,t-1}}{P_{I,t-1}} \right) = \sum_{i=1}^{N} X_i R_{i,t} \tag{7}$$

where

$$R_{i,t} = \frac{P_{I,T}}{P_{i,T}} \left( \frac{P_{i,t} - P_{i,t-1}}{P_{I,t-1}} \right) \tag{8}$$

Through substitution of (7) in (1) the function of objectives becomes:

$$TE = \sum_{t=1}^{T} \left( r_{I,t} - \sum_{i=1}^{N} X_i R_{i,t} \right)^2 \approx \sum_{t=1}^{T} r_{I,t}^2 - 2 \sum_{i=1}^{N} X_i \sum_{t=1}^{T} r_{I,t} R_{i,t} + \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sum_{t=1}^{T} R_{i,t} R_{j,t} \tag{9}$$

The first term in Equation (9) does not depend on the decision variables and is thus left out from now on. This function of objectives is a quadratic function of the decision variables.

## PRACTICAL TRADING RESTRICTIONS

An index portfolio must for a variety of reasons comply with numerous restrictions in practice, which means its constitution will have to diverge from the constitution of the index. Investors will select a number of stocks in advance, to constitute their index portfolios in order to comply with these restrictions. Even with this preselection, the investor has to decide what weights to attach to the selected stocks.

We impose three restrictions on this choice of weights:

$$\sum_{i=1}^{N} X_i = 1 \tag{10}$$

$$\forall i : X_i \geq \alpha_i \tag{11}$$

$$\forall i : X_i \leq \beta \quad \text{for} \quad (\beta_i \geq \alpha_i) \tag{12}$$

The first restriction makes the sum of the decision variables equal 1. They are, after all, weights. The second and third restrictions set the upper and the lower limits for the weight of stock *i*. These restrictions can be used, among other things, to limit the transaction costs when an established index portfolio is rearranged to a new index portfolio. We return to this later, but for now we set $\alpha_i = 0$ and $\beta_i = 1$. We exclude the possibility of going short because practice has shown this can lead to unstable portfolios. Beasley, Meade, and Chang [1999] show this by comparing it with the issue of optimizing portfolios, where it is clear that restrictions are needed to prevent unstable portfolios.

Minimizing the tracking error of Equation (9) under the linear restrictions is a quadratic programming problem for which several solution methods have been developed. In our case, we can find an optimal solution using the solver MOSEK.

## SELECTING SUBSET OF STOCKS

We can integrate the problem of pre-selection in the quadratic programming problem when we include an additional (binary) decision variable for each stock that indicates whether this stock will or will not be selected for the index portfolio. An additional restriction has to be added to prevent there being too many selected stocks, so that at most $Q$ stocks have a non-zero weight.

When we add the binary decision variables, the feasible domain is no longer convex. This seriously complicates the problem. One way to address this is to solve the problem for every combination of Q stocks, and to choose the combination with the lowest tracking error, but the calculations for this method take too much time for a problem of a realistic size (e.g., 600 available stocks and an index portfolio that can consist of 250 stocks). We can reduce this calculation time substantially with the branch and bounds method, but this is not sufficient for our purpose.

An alternative heuristic approach is to divide the problem into two subproblems. First, we use certain criteria to choose the Q stocks to be used. We then determine the optimal weights as described before. For the first subproblem, we test several different heuristics.

The first possibility is to select stocks so that as high as possible a part of the index value is also represented in the index portfolio. This is achieved by selecting the Q stocks with the highest market capitalization. Although the return on the index is also determined by stocks with a low market cap and a strong positive or negative return, this method will ignore these stocks. The solution selects the Q stocks with the highest value for the product of market capitalization and average absolute return.

A second option is to select the particular $Q$ stocks whose return is most correlated with the return on the index (although this says nothing about the strength of reaction to market information). The counterpart of this heuristic is to select the $Q$ stocks that instead react with average strength to market information (beta can also be greater than 1; we need as many stocks as possible with beta 1). These are the stocks for which the absolute difference between beta and 1 is the lowest, where beta is defined as the quotient of the covariance of the return of a stock with the return on the index and the variance of the return on the index.

Another alternative to finding $Q$ stocks is to select the particular stocks that are correlated as little as possible (we know it is not really advisable to include two very strong correlating stocks in a portfolio). This approach can be implemented by selecting the stocks with the strongest mutual correlation and by removing the stock with the lowest market capitalization until there are $Q$ stocks left. It is also possible to determine the average correlation with the other stocks that have not yet been removed and remove the stock with the highest average.

Selection on the basis of weights can also be made. This implies that the index portfolio is composed of all $N$ stocks, and the $Q$ stocks with the greatest weights are selected. On account of their low weights, the omitted stocks can contribute the least to the price of the sample index portfolio.

This method, however, may lead to poor solutions, such as if there is a sector or region with many stocks all strongly correlated with each other and on the other hand weakly correlated with the other stocks. Because of the great number of stocks in this sector, all the stocks involved will probably have low weights in the sample index portfolio. Thus a whole sector or region may be eliminated, which is undesirable when that sector or region has considerable representation in the index. This can be resolved by repeatedly removing a small number of stocks in several steps and thereby making a new sample index portfolio. It is possible both to remove an equal number of stocks every time and to remove a declining number, keeping the fraction of the number removed in every step constant.

We can use these various heuristics to construct other (better) index portfolios using a combination of approaches. A more sophisticated heuristic can lead to better results; see Fijn van Draat [2000].

## ADDING TRANSACTION COSTS

It is not desirable to have a portfolio with low tracking error and high transaction costs. The models we have described are particularly appropriate for *constructing* an index portfolio. We now want to *rebalance* an index portfolio, such as we would if the composition of the index changes. It is advisable to take transaction costs into account to prevent the whole portfolio from being changed with every rebalancing.

We add $N$ extra decision variables in order to include transaction costs correctly in the model:

$d_i$ = upper limit of the absolute change in weight of stock $i$ (for $i = 1, ..., N$).

Added input variables are ($T$ is the time just before the rebalance):

$\kappa_i$ = transaction costs for stock $i$ as a percentage of the transacted value (for $I = 1, ..., N$);
$K$ = maximum of transaction costs as a percentage of the value of the portfolio; and
$Y_i$ = fraction of the value of the portfolio that is invested in stock $i$ at time $T$ (for $i = 1, ..., N$).

Three new restrictions are also required:

$$\forall i : X_i - Y_i \leq d_i \tag{13}$$

$$\forall i : Y_i - X_i \leq d_i \tag{14}$$

$$\sum_{i=1}^{N} d_i \kappa_i \leq K \tag{15}$$

The first two restrictions cause the decision variable $d_i$ to be at least equal to the absolute change in weight of stock $i$. The third keeps the transaction costs under control.

## TESTING THE MODEL

To see whether the model works well, we use it to track the MSCI Europe Index. The number of stocks in this index varies, but moves in the range of 550–600 stocks. We backtest 27 investment strategies or runs for one year from July 2004 through June 2005. We also use a rolling time window of one year for the use of historical data, starting July 2003.

Before testing the entire model, it is useful to establish whether the approximation error made for the tracking error in Equation (9) by applying Equation (6) is not too big. Otherwise the model will be useless. It appears that the error may amount to 10% but that it is limited to a few percent at most with a 250-day history and using the selection-on-weights heuristics for the choice of stocks.

For feeding the model with historical data, a constant composition of the index is desired; yet the index changed regularly during our sample period. Therefore we first calculate a trial course of the index on the basis of all 250 days of prices, starting from the current composition.

By construction in this period, this portfolio has low tracking error, also referred to as in-sample tracking error. To see whether this index portfolio will also do well afterward, we virtually purchase the suggested portfolio in order to calculate the so-called out-of-sample (or real) tracking error. Backtesting started on July 1, 2004. Different optimization moments were defined, depending on the investment strategy, e.g., optimize or rebalance every last business day of the month. After running the model we made the suggested transactions, and then moved on to the next rebalancing moment. This was done until the last day of the backtesting period June 29, 2005.

Exhibit 1 shows the runs or investment strategies investigated. Stocks are selected on the basis of market capitalization. All tracking errors (TE) are arithmetic in basis points (1 bp = 0.01%) and annualized; the three-month tracking error is defined on a rolling time window. All trading budgets are annual budgets for the complete run.

## Exhibit 1

**Statistics for 27 Runs**

| | run 1 | run 2 | run 3 | run 4 | run 5 | run 6 | run 7 | run 8 | run 9 |
|---|---|---|---|---|---|---|---|---|---|
| Trading | 10% | 15% | 20% | 10% | 15% | 20% | 10% | 15% | 20% |
| Stocks in portfolio | 300 | 300 | 300 | 225 | 225 | 225 | 150 | 150 | 150 |
| Rebalance moments | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| TE total period | 35.26 | 33.63 | 37.00 | 45.97 | 43.49 | 44.73 | 70.29 | 68.32 | 67.74 |
| Mean 3 month TE | 34.36 | 33.26 | 35.80 | 44.92 | 42.22 | 43.40 | 7082 | 67.84 | 66.49 |
| Stddev 3 month TE | 2.245 | 3.027 | 3.543 | 3.95 | 3.452 | 4.719 | 4.143 | 4.887 | 5.595 |
| Period excess return | -0.91% | -0.49% | -0.81% | -1.13% | -1.37% | -1.41% | -1.79% | -1.77% | -1.70% |
| | | | | | | | | | |
| | run 10 | run 11 | run 12 | run 13 | run 14 | run 15 | run 16 | run 17 | run 18 |
| Trading | 10% | 15% | 20% | 10% | 15% | 20% | 10% | 15% | 20% |
| Stocks in portfolio | 300 | 300 | 300 | 225 | 225 | 225 | 150 | 150 | 150 |
| Rebalance moments | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| TE total period | 36.99 | 38.53 | 39.76 | 46.50 | 46.27 | 46.42 | 70.16 | 66.47 | 64.27 |
| Mean 3 month TE | 36.09 | 37.61 | 38.73 | 45.65 | 45.47 | 45.80 | 70.06 | 65.82 | 62.89 |
| Stddev 3 month TE | 3.299 | 4.337 | 5.046 | 3.23 | 4.056 | 4.154 | 5.778 | 4.231 | 4.855 |
| Period excess return | -0.73% | -0.75% | -0.74% | -1.33% | -1.30% | -1.19% | -1.82% | -1.80% | -1.69% |
| | | | | | | | | | |
| | run 19 | run 20 | run 21 | run 22 | run 23 | run 24 | run 25 | run 26 | run 27 |
| Trading | 10% | 15% | 20% | 10% | 15% | 20% | 10% | 15% | 20% |
| Stocks in portfolio | 300 | 300 | 300 | 225 | 225 | 225 | 150 | 150 | 150 |
| Rebalance moments | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| TE total period | 36.12 | 37.12 | 37.12 | 37.12 | 36.12 | 36.12 | 36.12 | 37.12 | 37.12 |
| Mean 3 month TE | 35.12 | 36.18 | 36.18 | 36.18 | 35.12 | 35.12 | 35.12 | 36.18 | 36.18 |
| Stddev 3 month TE | 2.638 | 3.411 | 3.411 | 3.411 | 2.638 | 2.638 | 2.638 | 3.411 | 3.411 |
| Period excess return | -1.04% | -1.07% | -1.07% | -1.07% | -1.04% | -1.04% | -1.04% | -1.07% | -1.07% |

We use three variants in rebalancing moments: 1) quarterly (MSCI rebalancing dates) plus an exceptional index change (i.e., on the date a major stock left the index); 2) on major changes (July 30, 2004, and the quarterly rebalancing date at the end of February); and 3) monthly (the last working day of the month). Note that all index changes are known at least one working day in advance, so all backtest runs would be valid and realistic investment strategies.

The results immediately show that having more stocks in the portfolio leads to lower tracking error. It is also obvious that when the rebalancing frequency is set to monthly, the other variables, trading and the number of stocks, influence the results very little (see Exhibit 2). In general there is also a relation between trading and tracking error; more trading budget leads to lower tracking error.

As Exhibit 3 shows, this relation is not as strong as one would initially expect. Further analysis shows that the size of the budget for trading should be in line with the amount of trading needed to execute the given investment strategy. Runs 3, 12, and 21, for example, have a lot of stocks (300) and a large trading budget (20%). The index weight of the stock with the smallest market capitalization is around 6 basis points. In these cases, there is a surplus in the trading budget.

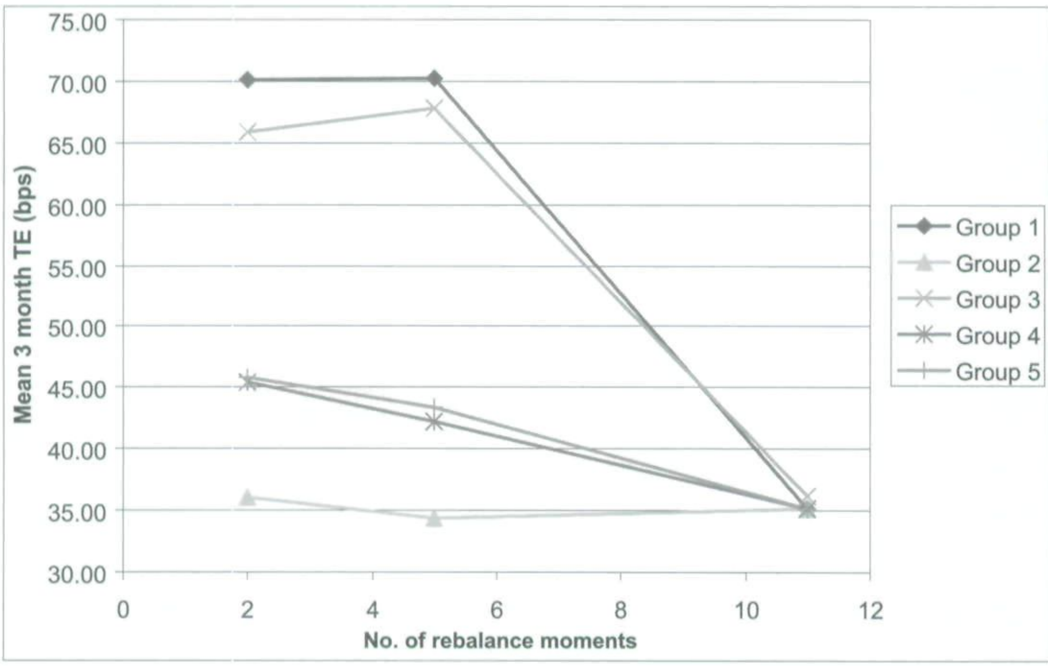
Exhibit 4 graphs the development of the values of the MSCI Europe Index and two optimized index portfolios (runs 7 and 25) using a rolling time window of one year for historical data, starting from one year before the period of interest. It turns out that our optimized index portfolios perform very similarly to the index. In all runs we have described, you can see that they underperformed the index. In the period we investigated, the smaller companies turned in significantly better returns than the large companies. As the pre-selection of securities in our picks model only the larger companies, the model selects those companies with on average lower returns than the benchmark.

## CONCLUSION

It is impossible for an index portfolio to move in perfect synchronicity with the index when there are restrictions on its composition. Yet there are several ways to express the tracking error of the index in a function that is free of restrictions. In our model we use the sum of the squared differences in return.

## EXHIBIT 2

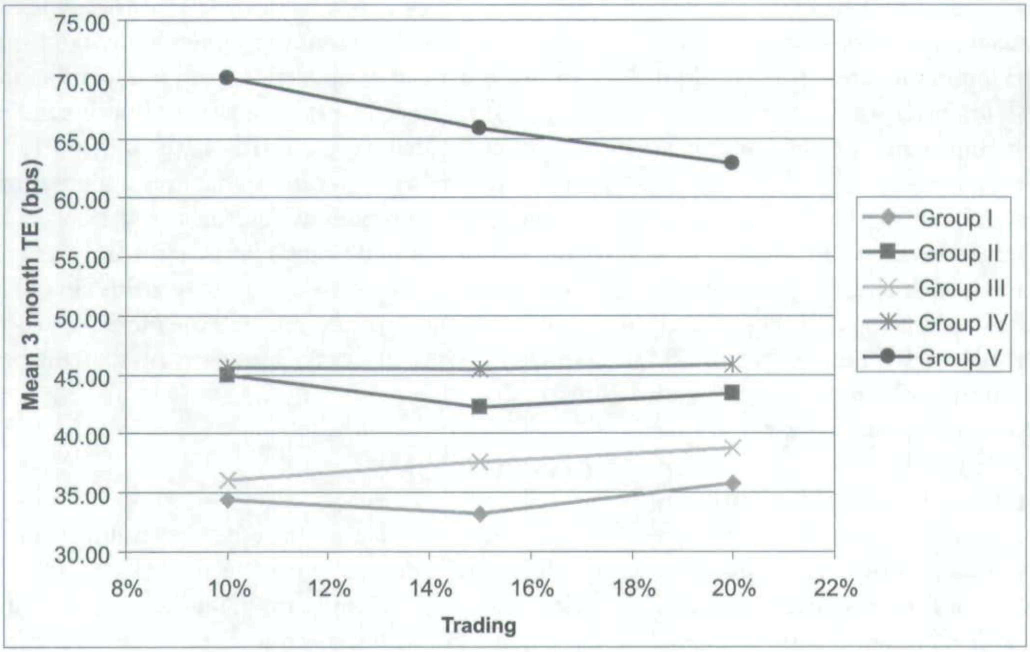
**Rebalancing Moments versus Tracking Error**




Groups constructed as follows:

| Group Number | Run Number |
|---|---|
| 1 | 16 |
| | 7 |
| | 25 |
| 2 | 13 |
| | 1 |
| | 19 |
| 3 | 17 |
| | 8 |
| | 26 |
| 4 | 14 |
| | 5 |
| | 23 |
| 5 | 15 |
| | 6 |
| | 24 |

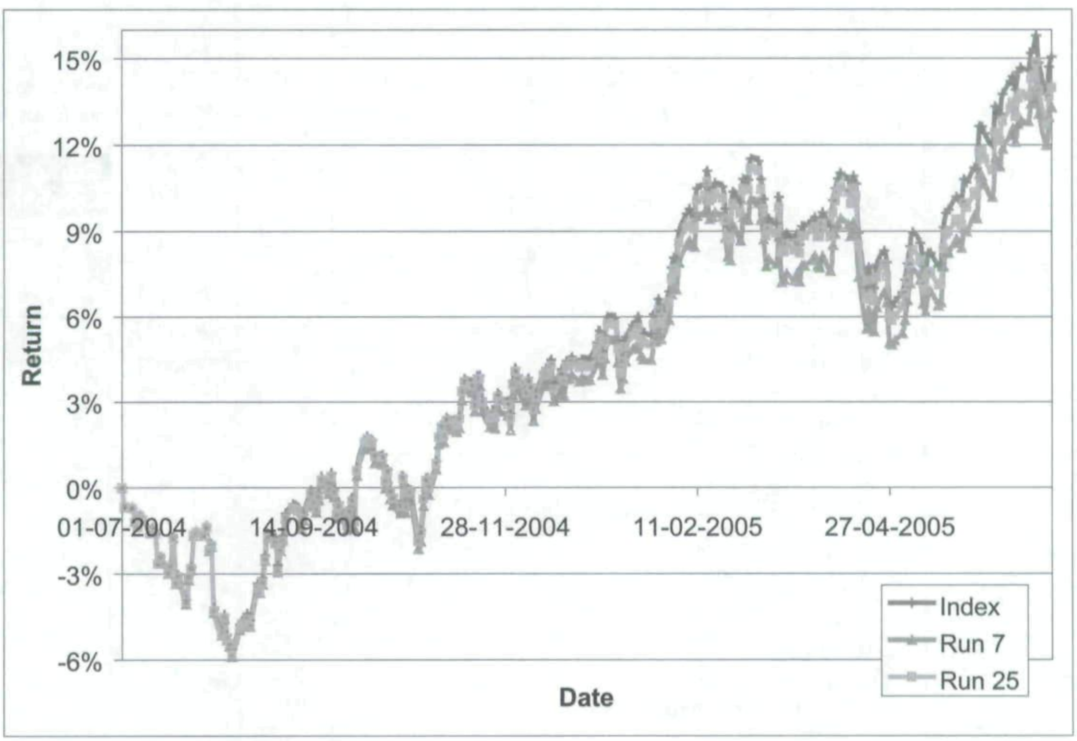
# EXHIBIT 3
**Trading versus Tracking Error**




Groups constructed as follows:

| Group Number | Run Number |
|---|---|
| I | 1 |
| | 2 |
| | 3 |
| II | 4 |
| | 5 |
| | 6 |
| III | 10 |
| | 11 |
| | 12 |
| IV | 13 |
| | 14 |
| | 15 |
| V | 16 |
| | 17 |
| | 18 |

# EXHIBIT 4
**Cumulative Returns**



We have developed a number of heuristics for constructing the index portfolio, because too much calculation time is required to get exact solutions. The first step is the selection of the stocks for the index portfolio, and then the weights are calculated for this selection.

Not every selection heuristic appears to be successful. The only selection heuristics that led to a reasonable out-of-sample tracking error are based on market capitalization, market capitalization and return, and weights with different numbers of steps.

In general, we could say that more stocks in the index portfolio are associated with lower tracking error. The same applies to trading; when the available trading budget is in line with trades needed to execute the investment strategy, a higher trading budget leads to lower tracking error. Even more important, very frequent rebalancing, monthly in our case, with a relatively low trading budget per rebalance moment, leads to lower tracking error. The monthly rebalancing frequency also outweighs the other determinants investigated, trading and number of stocks, resulting in stable tracking errors over time.

## ENDNOTE

The authors thank Petter Kolm for helpful comments.

## REFERENCES

Alexander, C., and A. Dimitriu. "A Comparison of Cointegration and Tracking Error Models for Mutual Funds and Hedge Funds." ISMA Centre Discussion Papers in Finance, 2004.

Beasley, J.E., N. Meade, and T.-J. Chang. "Index Tracking." The Management School, Imperial College, London, 1999.

Bertsimas, D., C. Darnell, and R. Soucy. "Portfolio Construction Through Mixed Integer Programming at Grantham, Mayo, Van Otterloo and Company." *Interfaces*, 29 (1999), pp. 49–66.

Clarke, R.C., S. Krase, and M. Statman. "Tracking Errors, Regret, and Tactical Asset Allocation." *The Journal of Portfolio Management*, 20 (Spring 1994), pp. 16–24.

Dynkin, L., J. Hyman, and W. Wu. "Replicating Index Returns with Treasury Futures." Research Report, 1997.

Fijn van Draat, L.M. "Index Tracking with Fund Selection and Transaction Costs." Research Report, Free University Amsterdam, 2000.

Focardi, S.M., and F.J. Fabozzi. "A Methodology for Index Tracking Based on Time Series Clustering." *Quantitative Finance*, 4 (August 2004), pp. 417–425.

Frino, A., D. Gallagher, T. Oetomo, and A. Neubert. "Index Design and Implications for Index Tracking: Evidence from S&P 500 Index Funds." *The Journal of Portfolio Management*, 30 (2004), pp. 89–96.

Jansen, R., and R. Van Dijk. "Optimal Benchmark Tracking with Small Portfolios." *The Journal of Portfolio Management*, 29 (Winter 2002), pp. 33–39.

Kritzman, M.P. "Incentive Fees: Some Problems and Some Solutions." *Financial Analysts Journal*, 43 (January/February 1987), pp. 21–26.

Larsen, G.A., G. Bruce, and G. Resnick. "Empirical Insights on Indexing." *The Journal of Portfolio Management*, 25 (Fall 1998), pp. 51–60.

Meade, N., and G.R. Salkin. "Index Funds—Construction and Performance Measurement." *Journal of the Operational Research Society*, 40 (1989), pp. 871–879.

Roll, R. "A Mean/Variance Analysis of Tracking Error." *The Journal of Portfolio Management*, 18 (Summer 1992), pp. 13–22.

Rudd, A. "Optimal Selection of Passive Portfolios." *Financial Management* (Spring 1980), pp. 57–66.

Rudolf, M., H.J. Wolter, and H. Zimmermann. "A Linear Model for Tracking Error Minimization." *Journal of Banking & Finance*, 23 (1999), pp. 85–103.

Speranza, M.G. "Linear Programming Models for Portfolio Optimization." *Finance*, 14 (1993), pp. 107–123.

Tabata, Y., and E. Takeda. "Bicriteria Optimization Problem of Designing an Index Fund." *Journal of the Operations Research Society*, 46 (1995), pp. 1023–1032.